COMBINATORICS OF COXETER GROUPS

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KAIST, Daejeon, South Korea
May 15-16 and 22-23

Introduction

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Combinatorics

Algebra

Geometry

Topology

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- Geometry: groups generated by linear reflections, symmetry groups of regular polytopes...
- Algebra: Root systems of Lie algebras and Kać-Moody algebras and Weyl groups, the theory of BN pairs.
- Topology: theory of buildings, right-angled Coxeter groups.

Rough classification of Coxeter groups

1. Finite groups

These are precisely the finite groups of isometries of $\mathbb{R}^n$ generated by *orthogonal* reflections.

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A complete classification exists for both families.

Finite: A_{n-1}, B_n, D_n and $I_2(m), F_4, H_3, H_4, E_6, E_7, E_8$.
Affine: $\widetilde{A}_{n-1}, \widetilde{B}_n, \widetilde{C}_n, \widetilde{D}_n$ and $\widetilde{G}_2, \widetilde{F}_4, \widetilde{E}_6, \widetilde{E}_7, \widetilde{E}_8$.

This will be explained in the second week of lectures.

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3. All other Coxeter groups

These correspond to groups of linear transformations of $\mathbb{R}^n$ generated by reflections which are *not* orthogonal.

→ Study of sub families: crystallographic groups, right-angled groups, simply laced groups, hyperbolic groups, . . .

The symmetric group

Various Coxeter sides

The symmetric group S_n

Fix n a positive integer.

Definition The symmetric group S_n is the group of bijections from $\{1, \cdots, n\}$ to itself under composition.

Elements of S_n are called permutations on n elements.

The product $\pi\tau$ is the permutation $i \mapsto \pi(\tau(i))$.

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There are two standard ways to write down a permutation:

- The one line notation $\pi(1)\pi(2)\cdots\pi(n)$.

$$S_3 = \{123, 132, 213, 231, 312, 321\}$$

- Every permutation π can be written uniquely as a product of disjoint cycles of the form $(i, \pi(i), \pi^2(i), \ldots, \pi^k(i), \ldots)$

$$S_3 = \{(1)(2)(3), (1)(23), (12)(3), (123), (132), (13)(2)\}$$

S_n: combinatorics

The symmetric group S_n has $n!$ elements (Proof sketch: n choices for $\pi(1)$, then $n-1$ choices for $\pi(2) \neq \pi(1)$, etc...).

As usual in combinatorics, one is interested in **refined enumeration** with respect to some natural statistics on S_n.

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Definition A descent of π is an index $i < n$ such that $\pi(i) > \pi(i+1)$.
An inversion of π is a pair $1 \leq i < j \leq n$ such that $\pi(j) < \pi(i)$.

Example The permutation 315624 has 2 descents and 6 inversions.

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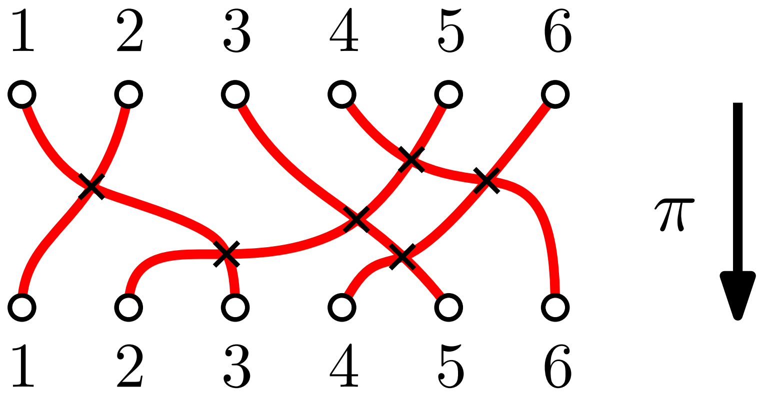
In the wiring diagram representation, inversions correspond to intersections of two curves, while descents correspond to intersections of two curves that started next to one another.



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Write $des(\sigma)$ for the number of descents of σ, $inv(\sigma)$ for the number of inversions of σ,

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σ	123	132	213	231	312	321
$des(\sigma)$	0	1	1	1	1	2
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We can write the generating polynomials

$$\sum_{\sigma\in S_3} q^{des(\sigma)} = 1 + 4q + q^2 \quad \text{and} \quad \sum_{\sigma\in S_3} q^{inv(\sigma)} = 1 + 2q + 2q^2 + q^3 = (1+q)(1+q+q^2)$$

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We will see that
- the statistics des and inv can be generalized to every Coxeter group
- the nice factorization occurring for inv can be generalized to every finite Coxeter group

S_n: algebra

The group S_n has the following presentation by generators and relations.

Generators: s_i for $i = 1, \ldots, n-1$
Relations: $s_i^2 = 1,\ s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1},\ s_i s_j = s_j s_i\ (j - i > 1)$.

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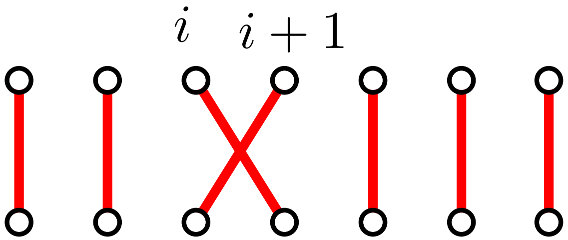

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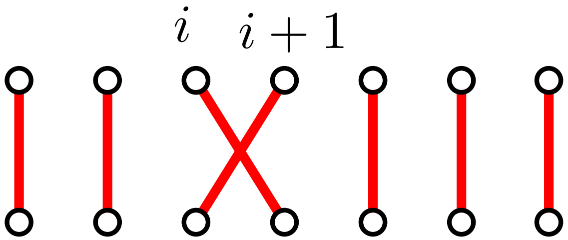

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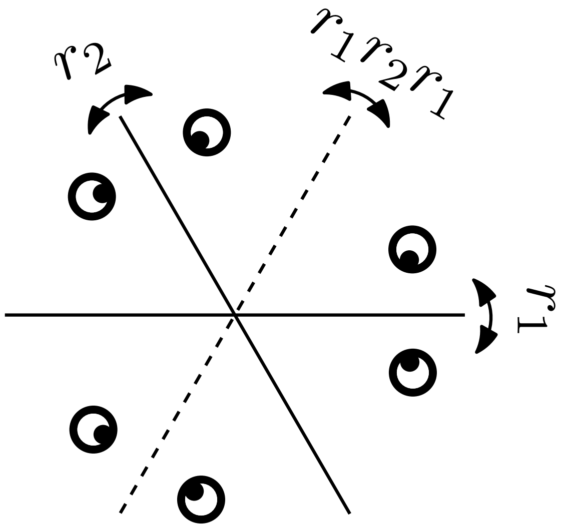
It is hard to work with the presentation of a group in general (many undecidable problems).

$\Rightarrow$ Coxeter groups in general will be defined by a certain presentation: luckily a certain geometric representation will help us discover their properties, and most natural questions are decidable.

S_n: geometry

The symmetric group can be represented by a certain subgroup of orthogonal transformations of the euclidean space $\mathbb{R}^n$.

Let r_i be the orthogonal reflection through $x_i = x_{i+1}$, for $i < n$. Then the group generated by the r_i is isomorphic to S_n.



Case of S^3: the picture is projected on the dimension 2 hyperplane $x_1 + x_2 + x_3 = 1$.

Notice also that there are $3!$ images of any point outside of reflecting hyperplanes.

Also S_n is the group of isometries preserving a regular simplex in $\mathbb{R}^{n-1}$

COXETER GROUPS

Definition

Coxeter matrix and diagram

Let S be a finite set.
Let also $M = (m_{st})_{s,t \in S}$ be a symmetric matrix such that:
(i) $m_{ss} = 1$ for all $s \in S$
(ii) $m_{st} \in \{2, 3, \ldots\} \cup \{\infty\}$ if $s \neq t$ in S.

$\Rightarrow$ This defines a Coxeter matrix.

Example $S = \{s_0, s_1, s_2\}$ $M = \begin{array}{c|ccc} & s_0 & s_1 & s_2 \\ s_0 & 1 & 4 & 2 \\ s_1 & 4 & 1 & 5 \\ s_2 & 2 & 5 & 1 \end{array}$

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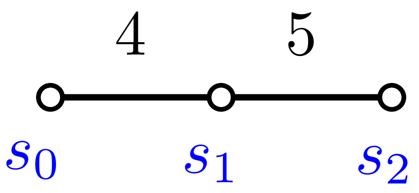
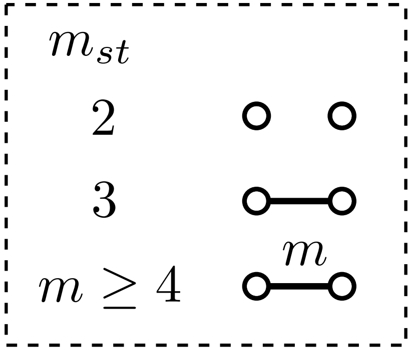
Coxeter diagram This is a labeled graph Γ encoding M. It has vertices S, an edge between s and t if $m_{st} \geq 3$, with label m_{st} when $m_{st} \geq 4$.





Coxeter group

Definition The Coxeter group W associated to M (or Γ) is defined by the following presentation:

- Generator set S
- Relations $(st)^{m_{st}} = 1$ for all $s, t \in S$ s.t. $m_{st} < \infty$.

Recall that $m_{ss} = 1$ by definition, which means that $s^2 = 1$ for all s.
$\Rightarrow$ W is generated by involutions.

Using this we can rewrite the relations for $s \neq t$, and we get the equivalent relations:

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Example For the previous M we get the group presented by

$$\boxed{\begin{gathered} s_0^2 = s_1^2 = s_2^2 = 1 \\ s_0 s_2 = s_2 s_0 \\ s_0 s_1 s_0 s_1 = s_1 s_0 s_1 s_0 \\ s_1 s_2 s_1 s_2 s_1 = s_2 s_1 s_2 s_1 s_2 \end{gathered}}$$

Coxeter group: examples

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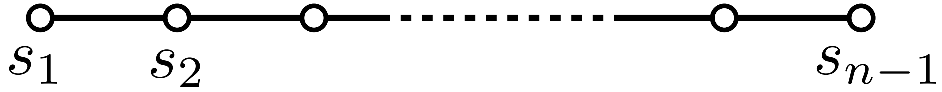

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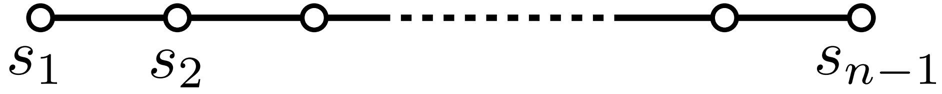

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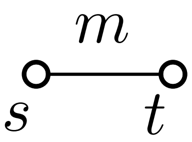

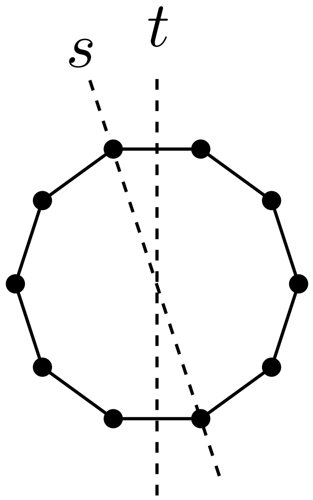

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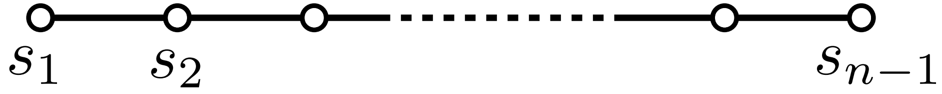

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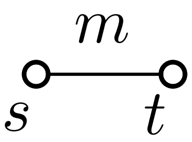
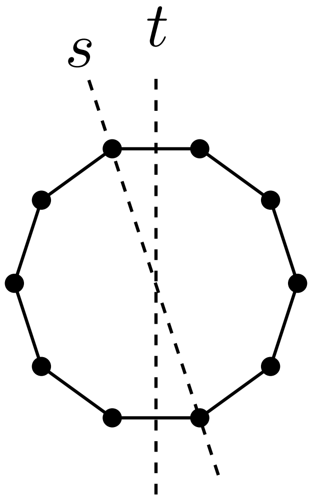
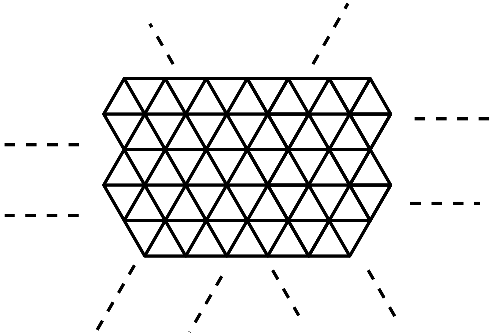




(3) Universal Coxeter group $m_{st} = \infty$ for all $s \neq t$.

(4) Affine isometries of  the tiling of the plane by equilateral triangles

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3 equivalent ways of stating the definition:

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- **Universal property** For any group G, and any function $f : S \to G$ satisfying $(f(s)f(t))^{m_{st}} = 1_G$ for all s, t s.t. $m_{st} < \infty$, there is a unique morphism $\tilde{f} : W \to G$ extending f.

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- **Combinatorial description** Let S^* be the free monoid on S, i.e. the set of words on the alphabet S with concatenation as product.
Consider the equivalence relation $\equiv$ generated by inserting or deleting the words $(st)^{m_{st}} = ststs \cdots (2m_{st}$ factors) for any $m_{st} < \infty$.
Then $S^*/\equiv$ is isomorphic to the group W.

Length

(W, S) is called a Coxeter system. In these lectures we will always assume that we are given such a system (and not just the group).

The cardinality $|S|$ is called the rank of the system. In fact the same group can be part of Coxeter systems of different ranks (Ex: $I_2(6)$).

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Definition Let w in W. Then the length $\ell(w)$ is the minimal k such that $w = s_1 s_2 \cdots s_k$.

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Definition A reduced expression for w is a word $s_1 \cdots s_k$ representing w with k minimal, i.e. where $k = \ell(w)$.

As we will see later in this course, a lot of the combinatorics of Coxeter groups comes from the study of reduced expressions.

The geometric representation of W

Geometric representation

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We also define a **bilinear form** $B : V \times V \to \mathbb{R}$ by

$$\boxed{B(\alpha_s, \alpha_t) := -\cos\left(\frac{\pi}{m_{st}}\right)} \qquad \text{(here } \tfrac{\pi}{\infty} = 0\text{)}$$

- B is symmetric, $B(\alpha_s, \alpha_s) = 1$.
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Anticipating on a future lecture, it turns out that B is positive definite (i.e. a scalar product) if and only if W is finite !

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For $s \in S$, define σ_s as the following linear transformation:

For $v \in V$, let $\sigma_s(v) := v - 2B(\alpha_s, v)\alpha_s$

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Proposition Let s, t in S.
(i) σ_s^2 is the identity on V;
(ii) $\sigma_s\sigma_t$ has order m_{st} in $GL(V)$ for $s \neq t$;
(iii) $B(\sigma_s(v), \sigma_s(v')) = B(v, v')$ for all $v, v' \in V$.

(i),(iii) are straightforward computations. Before sketching the proof of (ii), we record some immediate consequences:

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For $s \in S$, define σ_s as the following linear transformation:

For $v \in V$, let $\sigma_s(v) := v - 2B(\alpha_s, v)\alpha_s$

We see immediately that $\begin{cases} \sigma_s(\alpha_s) = -\alpha_s, \text{ and} \\ \sigma_s(v) = v \text{ whenever } B(\alpha_s, v) = 0. \end{cases}$

Proposition Let s, t in S.
(i) σ_s^2 is the identity on V;
(ii) $\sigma_s\sigma_t$ has order m_{st} in $GL(V)$ for $s \neq t$;
(iii) $B(\sigma_s(v), \sigma_s(v')) = B(v, v')$ for all $v, v' \in V$.

(i),(iii) are straightforward computations. Before sketching the proof of (ii), we record some immediate consequences:

Proposition The map $s \mapsto \sigma_s$ extends to a homomorphism $\sigma : W \to GL(V)$.

This morphism σ is the geometric representation of W.
As we will see soon, this representation is always *faithful*, meaning that the map σ is injective.

Geometric representation

Corollary The order of st in W is exactly m_{st}.

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Proof that $\sigma_s \sigma_t$ has order m_{st} (Sketch)

Write $V_{st} = \mathbb{R}\alpha_s \oplus \mathbb{R}\alpha_t$, and $V_{st}^{\perp} = \{v \in V \mid B(v, w) = 0$ for $w \in V_{st}$

Then one has $V = V_{st} \oplus V_{st}^{\perp}$ if $m_{st} \neq \infty$ (claim 1).

In this case, $\sigma_s \sigma_t$ acts trivially on $V_{st}^{\perp}$, while its acts on V_{st} by a rotation of order m_{st} as can be seen by computing its matrix in the canonical basis. (claim 2)

If $m_{st} = \infty$, then $(\sigma_s \sigma_t)^k(\alpha_s) = 2k(\alpha_s + \alpha_t) + \alpha_s$ for k integer,. (claim 3) so that $\sigma_s \sigma_t$ has infinite order. $\square$

Roots

For simplicity we will write $w \cdot v$ or wv instead of $\sigma(w)(v)$.

Definition The root system of (W, S) is $\Phi = \{w\alpha_s \mid w \in W, s \in S\}$.

• The elements of Φ are called roots, and will be usually denoted by the first greek letters $\alpha, \beta, \gamma, \ldots$
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• Since B is invariant by W, we have that for any root α, $B(\alpha, \alpha) = 1$: indeed $B(\alpha, \alpha) = B(w\alpha_s, w\alpha_s) = B(\alpha_s, \alpha_s) = 1$

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$\Rightarrow$ if $v = c\alpha$ is another root($c \in \mathbb{R}$), then $c = \pm 1$.

• Say that α is positive $(\alpha > 0)$ if $\alpha = \sum_s c_s \alpha_s$ with all $c_s \geq 0$, and negative $(\alpha < 0)$ if all $c_s \leq 0$

Let Φ^+ be the set of positive roots, Φ^- be the set of negative roots.

Roots and length

Theorem Let $w \in W, s \in S$.
(i) $\ell(ws) > \ell(w) \quad \Rightarrow \quad w \cdot \alpha_s > 0$
(ii) $\ell(ws) < \ell(w) \quad \Rightarrow \quad w \cdot \alpha_s < 0$

We admit this fundamental result, whose proof is fairly technical. Let us record the immediate consequences.

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Corollary 3 $s \in S$ sends α_s to $-\alpha_s$ and permutes the other positive roots. The length $\ell(w)$ is equal to the number of positive roots sent to negative roots by w.

Explicitly, if $s_1 \cdots s_k$ is a reduced expression for w, the set $N(w) = \Phi^+ \cap w^{-1}\Phi^-$ is given by the k distinct roots $s_k \cdots s_{i+1}\alpha_{s_i}$.

Further properties of Coxeter groups

Parabolic subgroups

- Let I be a subset of S, and W_I the subgroup of W generated by I.

Definition W_I is called a (standard) parabolic subgroup.

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Proposition (a) (W_I, I) is a Coxeter system with matrix $(m_{st})_{s,t \in I}$.
(b) For any $w \in W$ and reduced expression $w = s_1 \cdots s_k$, if $w \in W_I$ then all s_i are in I.

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Corollary The length function ℓ_I coincides with the restriction of the length function ℓ to W_I.

- Define W^I as the set of $w \in W$ such that $\ell(ws) > \ell(w)$ for all $s \in I$.

Proposition Fix $I \subseteq S$. For any $w \in W$, there exists unique elements $w_I \in W_I, w^I \in W^I$ such that $w = w^I w_I$. Then $\ell(w) = \ell(w^I) + \ell(w_I)$. Moreover w^I is the unique element of minimal length in the coset wW_I.

Enumeration with respect to length

We now want to compute the following generating function:

$$W(q) = \sum_{w \in W} q^{\ell(w)} \in \mathbb{N}[[q]]$$

This is well defined since for each $k \geq 0$ there are a finite number of elements of length k.

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$$\sum_{I \subseteq S} (-1)^{|I|} W^I(q) = \sum_{I \subseteq S} (-1)^{|I|} \frac{W(q)}{W_I(q)} = \begin{cases} 0 \text{ if } W \text{ is infinite} \\ q^{|\Phi^+|} \text{ if } W \text{ is finite} \end{cases}$$

The proof uses a little fact about finite Coxeter groups (to be seen in a later lecture).

Corollary $W(q)$ is an explicitly computable rational function.

Roots and reflections

One can extend the correspondence $\alpha_s \leftrightarrow s$ to any root as follows.

Given $w \in W, s \in S$ one checks that $wsw^{-1} \cdot v = v - 2B(\alpha, v)\alpha$, where α is the root $w(\alpha_s)$.

This only depends on α, so we can define $t_\alpha := wsw^{-1}$ (note $t_\alpha = t_{-\alpha}$). It acts by sending α to $-\alpha$ and fixing pointwise $\{v \in V \mid B(v, \alpha) = 0\}$

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Write $T = \cup_w wSw^{-1}$ for the set of reflections t_α.

Proposition $\alpha \mapsto t_\alpha$ is a bijection from Φ^+ to T.

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The following generalizes our previous theorem:

Proposition Let $w \in W, \alpha \in \Phi^+$. Then $\ell(wt_\alpha) > \ell(t)$ if and only if $w\alpha > 0$.

Deletion and Exchange properties

Proposition (Strong Exchange Property) Let $w = s_1 \cdots s_k$ be an expression for w (not necessarily reduced). If $\ell(wt) < \ell(w)$ for a reflection $t \in T$, then there exists $i \in \{1, \ldots, k\}$ such that $wt = s_1 \cdots \hat{s}_i \cdots s_k$.
If the expression is reduced then i is unique.

This is used most often when $t \in S$, in which case it is simply called the exchange condition.

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Corollary (Deletion Property) If $w = s_1 \cdots s_k$ is an expression for w which is not reduced, then there exists $i < j$ such that
$w = s_1 \cdots \hat{s}_i \cdots \hat{s}_j \cdots s_k$.

In particular any reduced expression can be obtained from a non reduced ones by deleting certain letters.

Characterization of Coxeter groups

Theorem Let W be a group, S a set of generators of order 2. Then the following are equivalent:
(i) (W, S) is a Coxeter system;
(ii) (W, S) has the Exchange Property;
(iii) (W, S) has the Deletion Property.

The hard part of the proof is to show (iii) $\Rightarrow$ (i).

The Coxeter matrix M is naturally defined by letting m_{st} be the order of st in W.

Weak Order and Reduced Expressions

Weak order: definition

Definition We write $u \leq_R w$ if there exist generators $s_i \in S$ such that $w = us_1 \cdots s_k$ and $\ell(us_1 \cdots s_i) = \ell(u) + i$ for all i

This defines the (right) weak order in W.

There is a one-to-one correspondence between reduced words for w and maximal $\leq_R$ chains from e to w.

Examples Dihedral groups, symmetric groups.

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- Define $Inv(w) := N(w^{-1}) = \Phi^+ \cap w(\Phi^-)$.

Proposition $u \leq_R v \Leftrightarrow Inv(u) \subseteq Inv(v)$

Therefore $w \mapsto Inv(w)$ is an order- and rank-preserving embedding from $(W, \leq_R)$ to $(\mathcal{P}_{finite}(\Phi^+), \subseteq)$.

The lattice property

- An element x in a poset is a meet of a subset A if $x \leq y$ for all $y \in A$ and if $u \leq y$ for all $y \in A$ then $u \leq x$.
- A poset where every subset has a meet is called a complete meet-semilattice.

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Theorem $(W, \leq_R)$ is a complete meet-semilattice.

The proof uses the following lemma:

> **Lemma** If $\ell(su) < \ell(u), \ell(sv) < \ell(v)$, then $u \leq_R v$ if and only if $su \leq_R sv$.

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Lemma If $\ell(su) < \ell(u), \ell(sv) < \ell(v)$, then $u \leq_R v$ if and only if $su \leq_R sv$.

• In a complete meet-semilattice, every subset A bounded from above possesses a join ($=$ the meet of all elements above A).
We will see that finite Coxeter groups possess a greatest element for $\leq_R$, noted w_0, hence:

Corollary If W is finite then the weak order is a lattice

The word property

Theorem [Matsumoto-Tits] Let $w \in W$.
(a) Any expression for w can be transformed into a reduced expression by using braid relations and nil moves $s^2 \to 1$.
(b) Any two reduced expressions for w can be connected by using only braid relations.

Note that this gives a (very inefficient) way to solve the word problem in W, i.e. recognize when two expressions represent the same word.

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Note that this gives a (very inefficient) way to solve the word problem in W, i.e. recognize when two expressions represent the same word.

- Given a word $\mathbf{w}$ in S^*, its support is the set of letters that appear in $\mathbf{w}$

Corollary Any two reduced expressions for an element $w \in W$ have the same support.

(As noticed by some participants, this can also be deduced from the results about reduced decompositions in parabolic subgroups)

The finite case

Finite Coxeter groups

By reflection in V, we mean an element of $GL(V)$ which sends a vector α to $-\alpha$ and fixes pointwise a given hyperplane H. It can be written as $x \mapsto x - 2\lambda(x)\alpha$ with $\lambda \in V^*$ satisfying $\lambda(\alpha) = 1$.
When V has a scalar product $(\cdot, \cdot)$, the reflection is orthogonal if H is the hyperplane orthogonal to α. In this case $\lambda = (\alpha, \cdot)/(\alpha, \alpha)$

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Proposition The group of isometries of a regular polytope is a finite reflection group.

A polytope P is the convex hull of a finite number of points in $\mathbb{R}^n$.
It is regular if its isometry group is transitive on the complete flags of P, i.e. its sequences of its faces $F_0 \subset F_1 \subset \cdots \subset F_{n-1}$ where F_i has dimension i.

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Theorem Let (W, S) be a Coxeter system. Then the following statements are equivalent:
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- (b) implies (c): One can show for any W that in the natural topology of $GL(V) \simeq GL_n(\mathbb{R}) \subset M_n(\mathbb{R}) \simeq \mathbb{R}^{n^2}$), $\sigma(W)$ is a discrete subgroup. Now if B is positive definite, then $GL(V)_B$ is compact, which forces $W \simeq \sigma(W)$ to be finite.

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This shows that among Coxeter groups, the finite ones are reflection groups. We now show that actually any finite reflection group is a Coxeter group.

Finite Reflection Groups are Coxeter Groups

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Plan of the proof

1. One attaches to W a certain root system Φ constructed from the set T of all reflections in W.
2. Φ contains positive systems Π and simple systems Δ, which determine one another.
3. Given Δ, Π, then t_α for $\alpha \in \Delta$ permutes roots in $\Pi - \{\alpha\}$.
4. Fix Δ from now on. The reflections in $S := \{t_\alpha, \alpha \in \Delta\}$ generate W.
5. Define $n(w) = \Pi \cap w^{-1}(-\Pi)$ as for Coxeter groups. Then $n(wt_\alpha) = n(w) + 1 \Rightarrow w(\alpha) \in \Pi$ holds, as well as the deletion property with the condition $n(w) < r$ instead of $\ell(w) < r$.
6. In fact $\forall w \in W, n(w) = \ell(w)$ which proves the theorem.

Remark Note how the constructions differ from Coxeter groups: positive roots, generators are not given here

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We will focus on the important constructions occurring at the beginning of the proof.

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Root systems

1. Root system

Let $\pm\alpha$ be the unit vectors determining the reflection $s_\alpha \in W$. Then the set Φ of these vectors satisfies (1) $\Phi \cap \mathbb{R}\alpha = \{\alpha, -\alpha\}$ and (2) $t_\alpha \Phi = \Phi$ for all $\alpha \in \Phi$.

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Associate to Φ the group $W(\Phi)$ generated by the $t_\alpha,\ \alpha \in \Phi$. $W(\Phi)$ is a reflection group, and all reflection groups are obtained in this way. We must thus prove that for any root system, $W(\Phi)$ is a Coxeter group..

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2. Positive and Simple systems

- A set $\Pi \subset \Phi$ is a positive system for Φ if it can be written $\Phi \cap \{v \in V \mid (v_0|v) > 0\}$ where $\{v \in V \mid (v_0|v) = 0\}$ intersects Φ trivially.
- A set $\Delta \subset \Phi$ is a simple system if it is linearly independent, has the same linear span as Φ, and all roots in Φ have their coefficients on Δ either all nonnegative or all nonpositive.

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• Now given Π, let Δ be the subset of roots which cannot be written as a positive linear combination of ≥ 2 roots in Π. Geometrically, they are the roots in Π generating the extreme rays of $C(\Phi)$.

We have to show that Δ is a simple system, that is, it is linearly independent. This is done by showing that $(\alpha, \beta) \leq 0$ for any $\alpha \neq \beta$ in Δ, any such set of vectors being automatically independent. □

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Given $\Delta \subset \Pi$, a consequence of the fact that t_α for $\alpha \in \Delta$ permutes $\Pi - \{\alpha\}$ is the following:

Proposition W acts transitively on positive systems.

The longest element

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This finally proves the lattice property. The Matsumoto-Tits property is a consequence of the following:

Proposition Let W' be any Coxeter group, and let $I \subseteq S$. Then I has a join in $(W', \leq_R)$ iff W'_I is finite, in which case the join is $w_0(I)$.

Enumeration

Let (W, S) be a finite Coxeter group of rank n. Let c be a Coxeter element of W, i.e. the product of generators in a certain order.

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(a) $h = |\Phi|/n$
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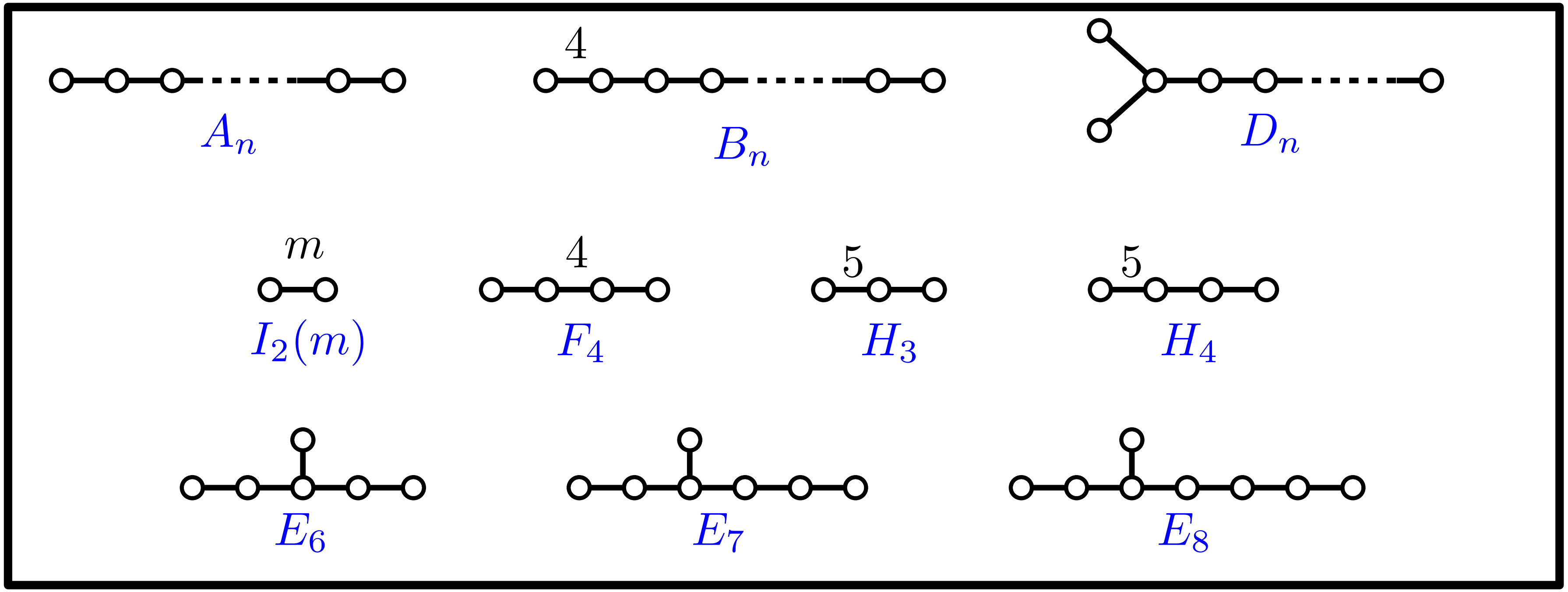
Example (continued) S_n corresponds to type A_{n-1}. Here the rank is $n-1$, $|\Phi| = n(n-1)$ and $h = n$.
The exponents e_i are $1, 2, \ldots, n-1$, which coincides with our previous enumeration.

Classification of finite Coxeter groups

Classification theorem Let W be a Coxeter group. Then W is finite iff every connected component of its Coxeter diagram $\Gamma(W)$ belongs to the following list:



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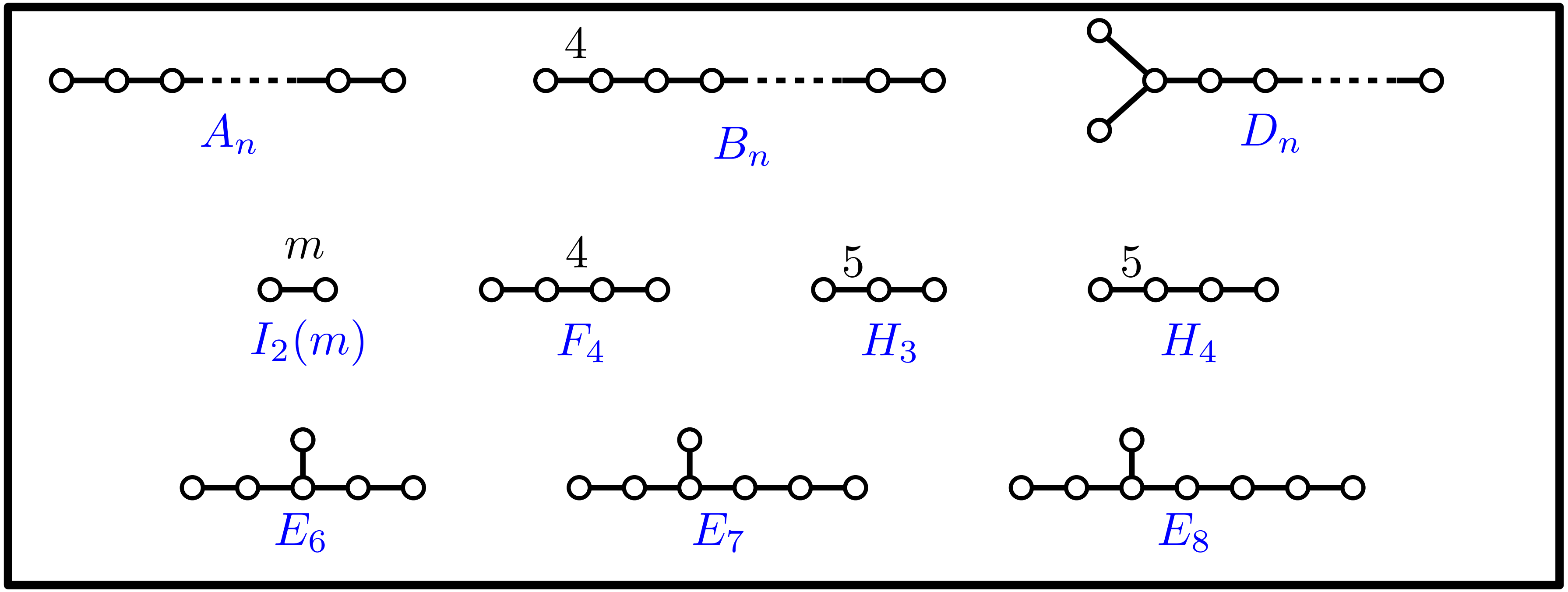

Strategy of the proof:

- Enough to assume diagram $\Gamma(W)$ connected (i.e. W is irreducible).
- $\Leftarrow$ Check that $B(\cdot,\cdot)$ is definite positive for each of these.
- $\Rightarrow$ Prove patiently that all other diagrams correspond to infinite groups, for instance by checking that $B(\cdot,\cdot)$ is not positive definite.

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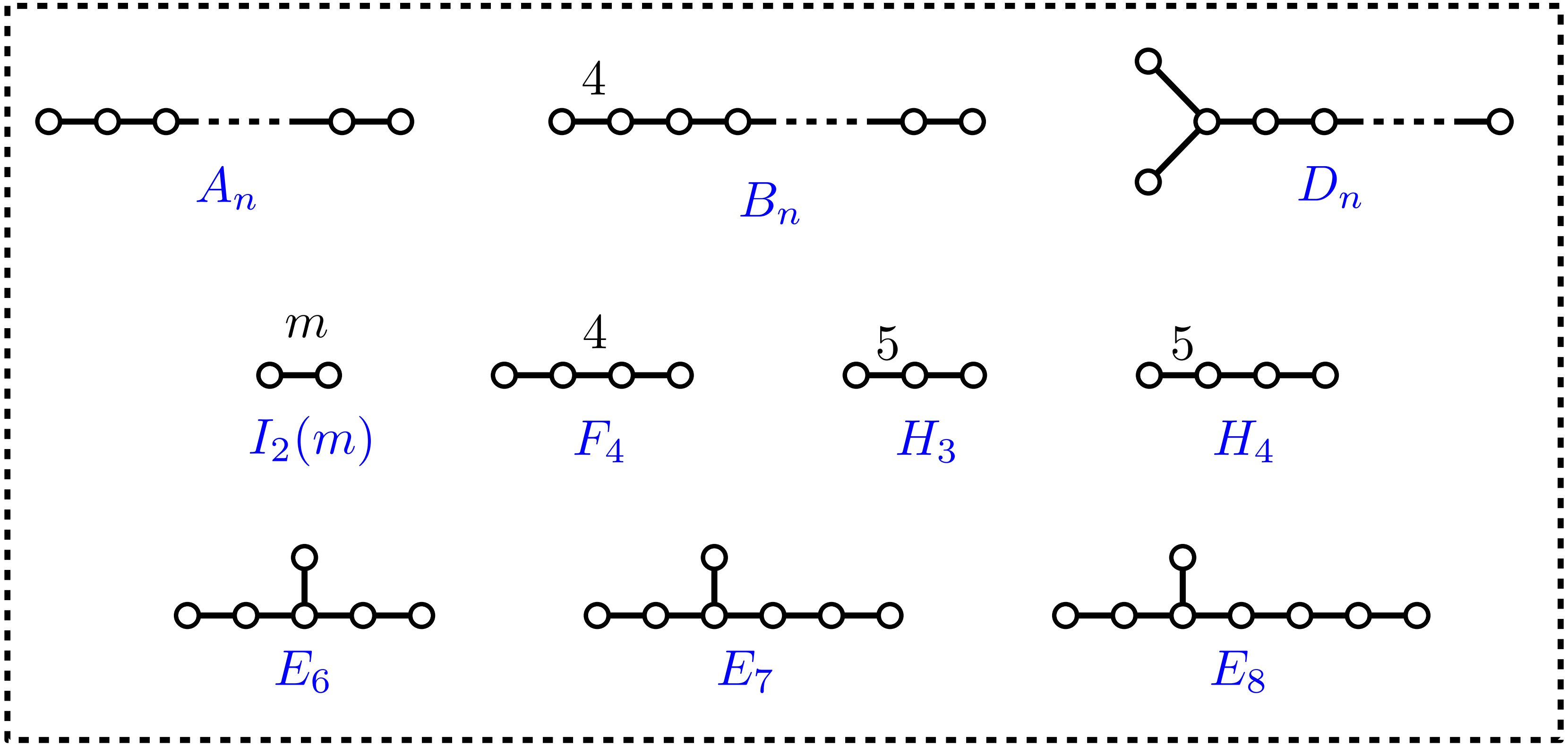
• So we first need to prove that for every diagram Γ in the list, the principal minors of the matrix of B are all positive.

• In the other direction, take any connected Γ, and show that it belongs to the list. For this prove successively:

1. Γ has all $m_{st} < \infty$
2. Γ has no cycle.
3. Γ has at most one edge $m_{st} > 3$.
4. If Γ has an edge $m_{st} > 3$, it is a linear diagram.
5. Γ has at most one branching point.
6. If Γ has a branching point, then it is of degree 3.

Three important subclasses



Among finite irreducible Coxeter groups, various subclasses occur naturally in various areas.

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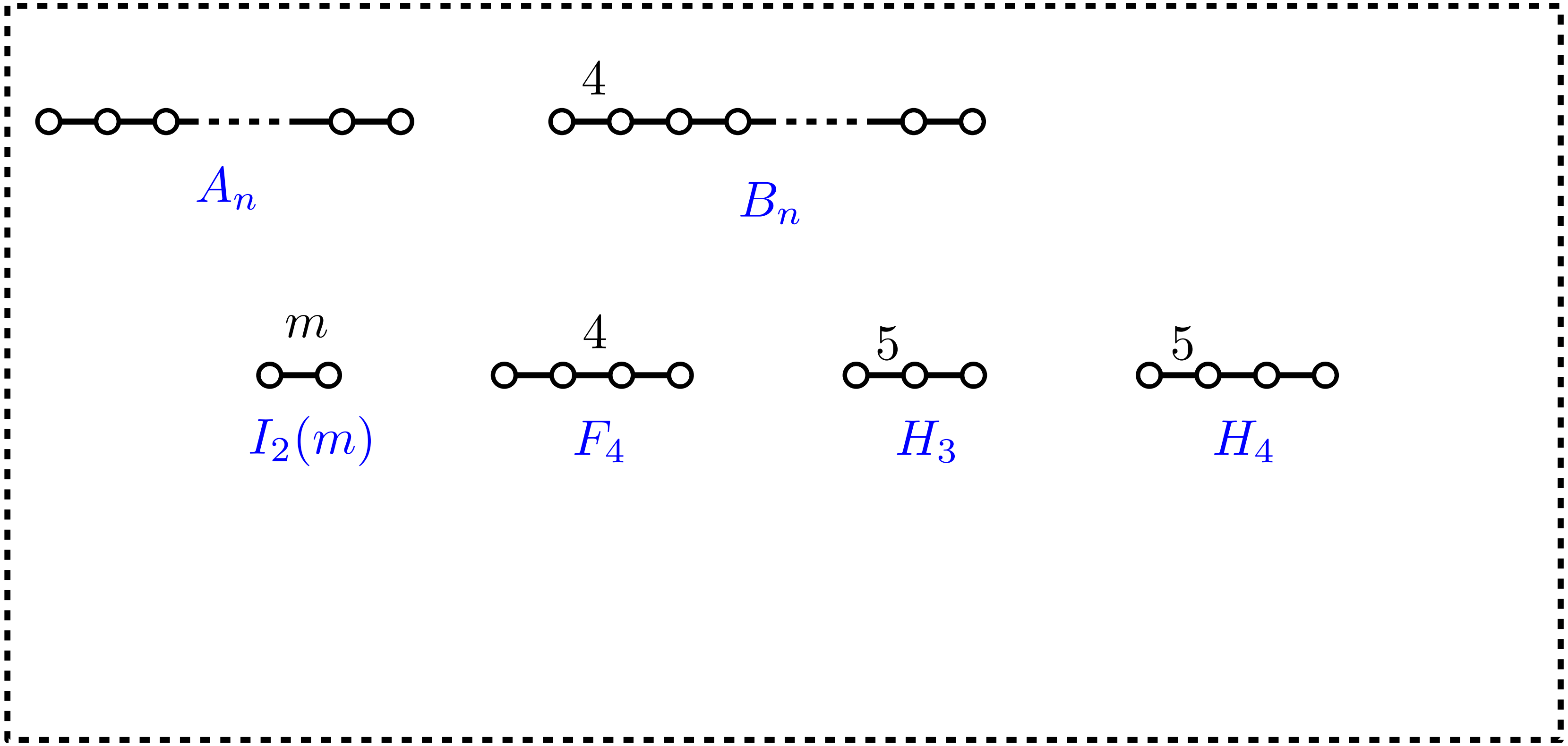

1. Isometries of regular polytopes: As already mentioned, these form finite reflection groups, so they have a Coxeter presentation.

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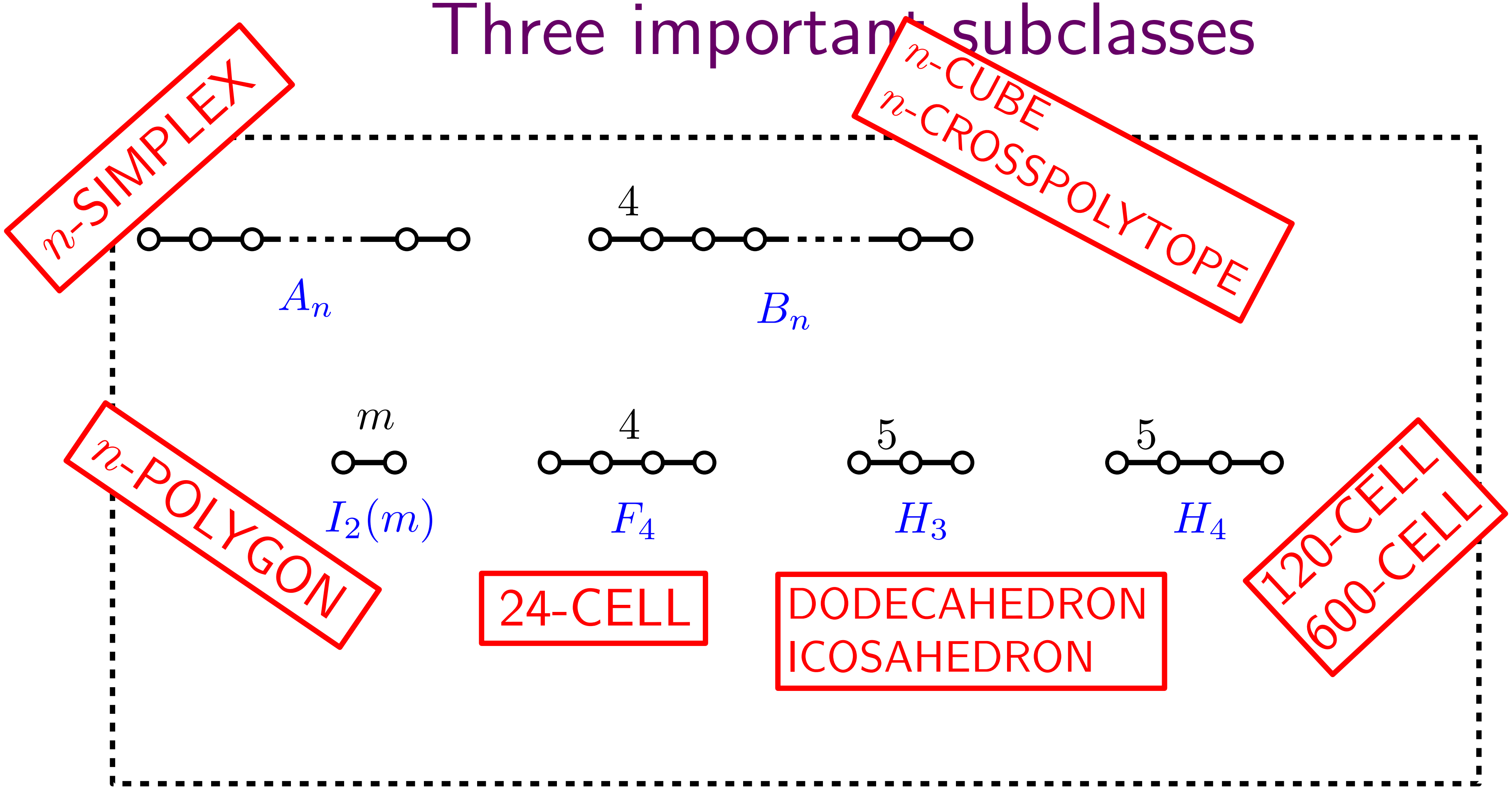
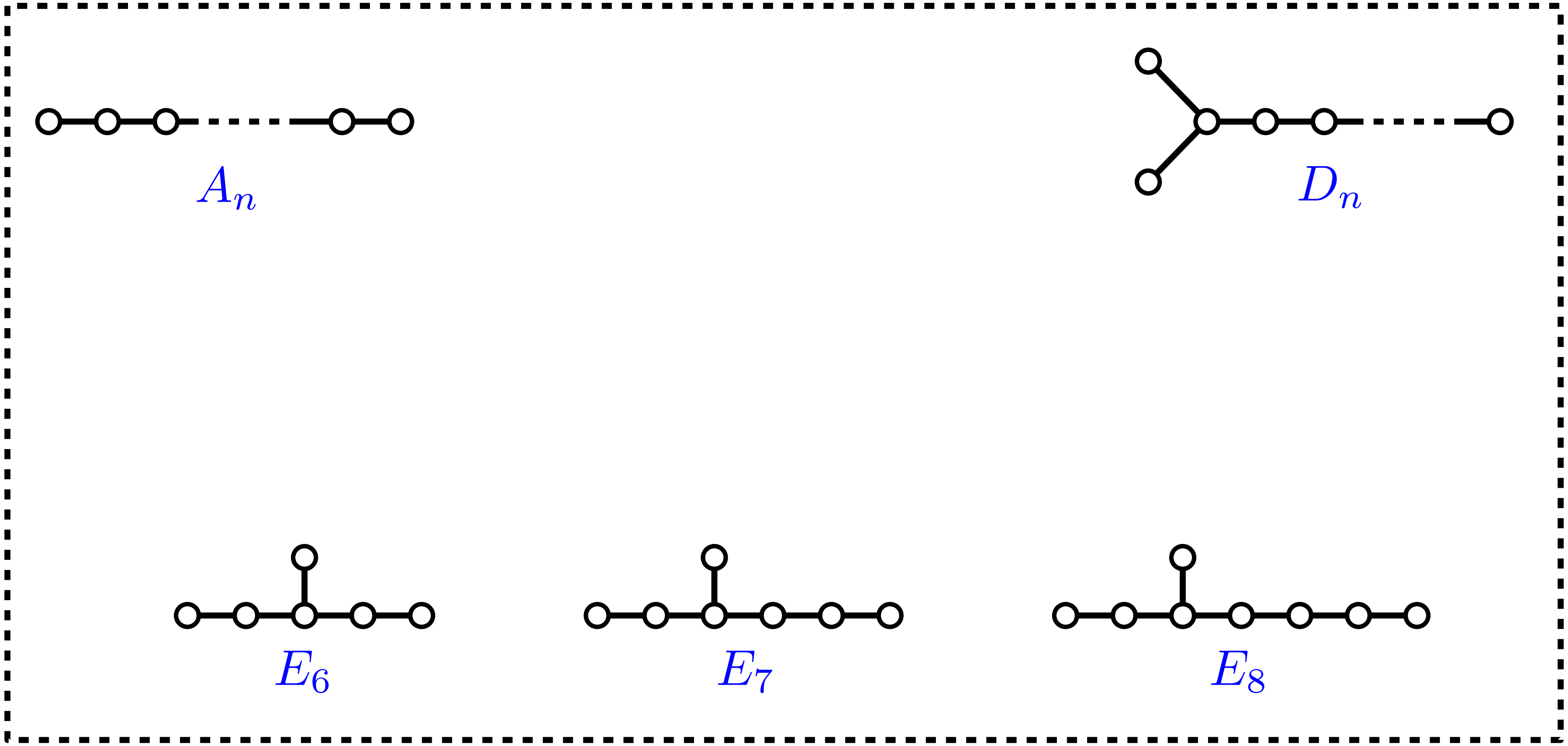


1. Isometries of regular polytopes: As already mentioned, these form finite reflection groups, so they have a Coxeter presentation.

In the classification, they correspond precisely to linear diagrams (no branching point)

$\Rightarrow$ Can essentially reconstruct the polytope from the diagram.

Three important subclasses



2. Simply-Laced case: At the level of Coxeter groups, this means that $m_{st} \leq 3$ for all s, t.

This forms the well-known ADE classification, ubiquitous in mathematics: simple Lie algebras, quivers of finite type (Gabriel's theorem), $\{0,1\}$-matrices of $||\cdot||_2$-norm smaller than $2\ldots$

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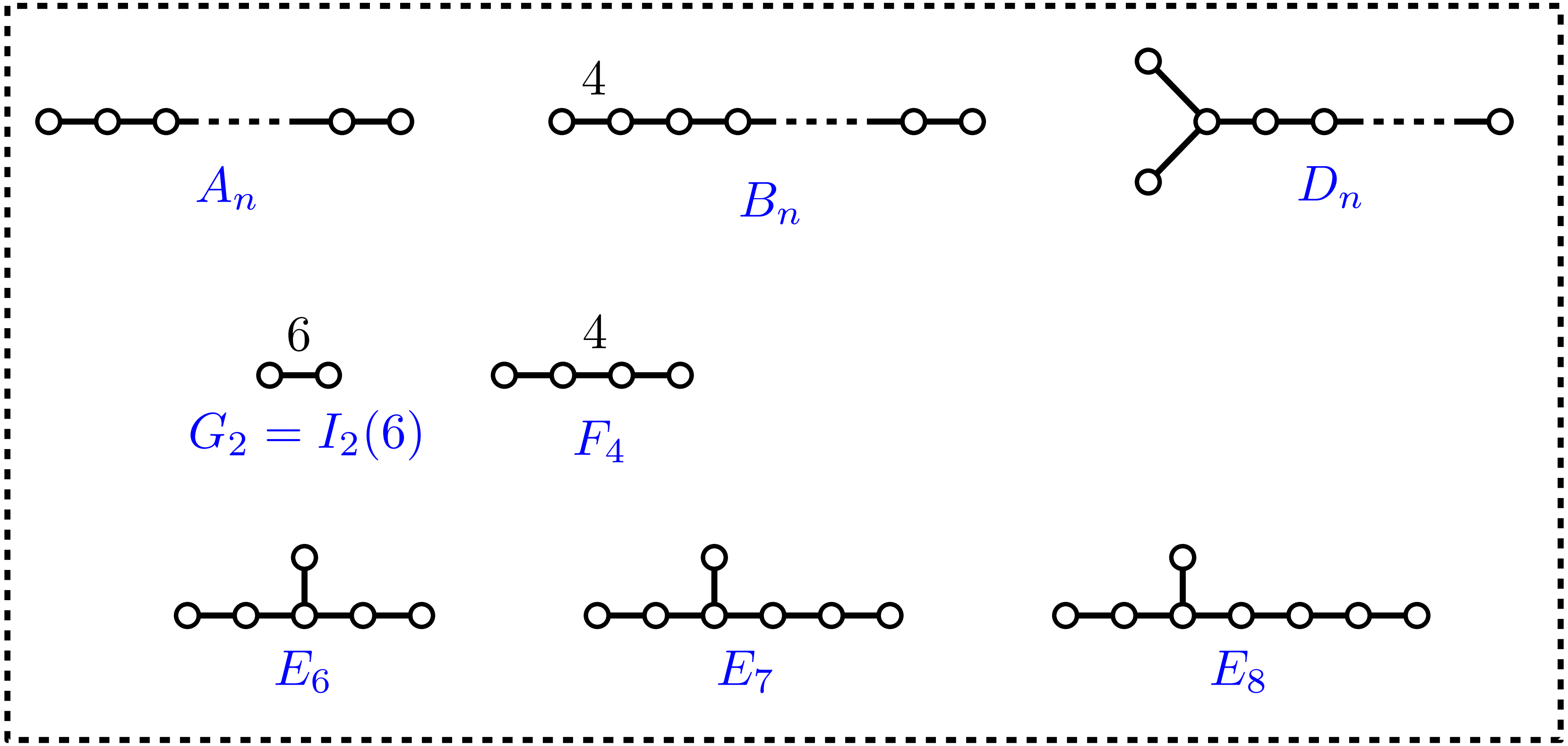

3. Crystallographic groups A group is called crystallographic if it leaves a lattice L invariant, while a finite root system Φ is called crystallographic if it satisfies $2(\alpha|\beta)/(\alpha/\alpha) \in \mathbb{Z}$ for any roots α, β.

The two notions coincide, the latter one being equivalent to $m_{st} \in \{2, 3, 4, 6\}$ for $s \neq t$ in the Coxeter group presentation.

Fundamental domain

Let W be a finite reflection group.

Fix $\Delta \subset \Pi$. Given $\alpha \in \Pi$ with orthogonal hyperplane H_α define the half spaces $H_\alpha^+ = \{v \in V \mid (\alpha|v) > 0\}$ and $H_\alpha^- = \{v \in V \mid (\alpha|v) < 0\}$

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Theorem D is a fundamental domain for the action of W on V: for any $v \in V$, there is a unique $v' \in D$ such that $w \cdot v = v'$ for a certain $w \in W$.

Moreover, if $w \cdot v = v'$ and $v, v' \in D$, then w is a product of simple reflections fixing λ.

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Consequence

- Since W acts simply transively on simple systems, then $w \mapsto wC$ gives a bijection from W to chambers, which are the connected components of $V - \{H_\alpha, \alpha \in \Pi\}$
- The number of hyperplanes separating C from wC is $\ell(w)$.

The affine case

Affine reflection groups

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Pick $\Delta \subset \Pi$ simple and positive systems for Φ. Then there exists a highest root $\widetilde{\alpha}$, such that $\alpha \geq \alpha'$ for any other root.

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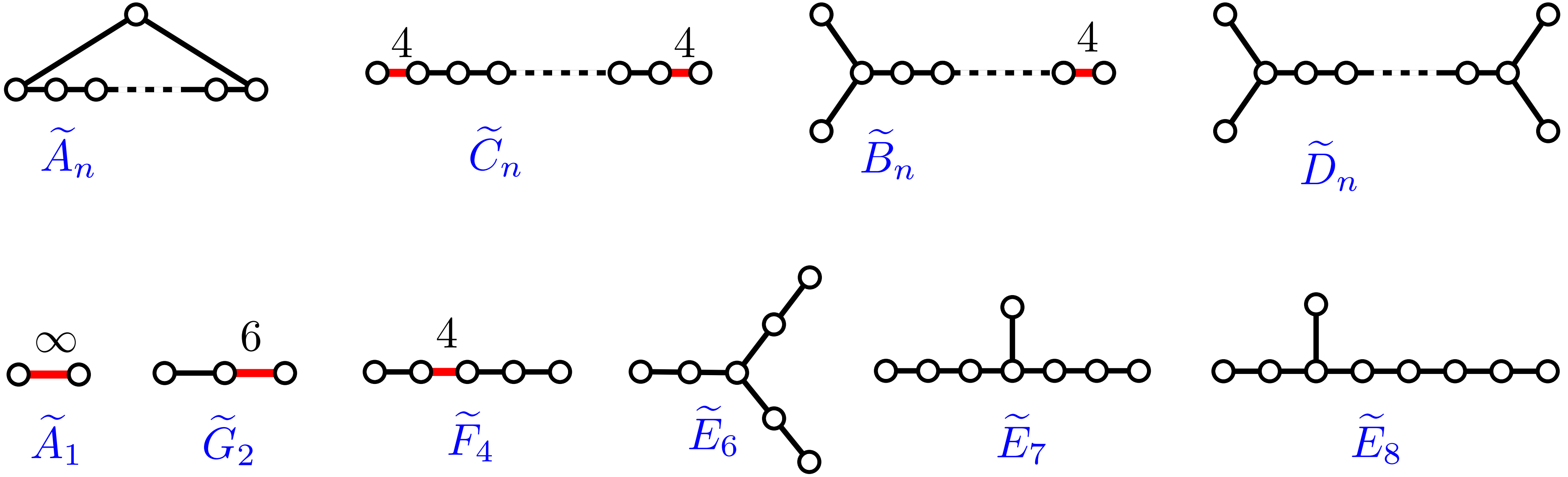
Remark There are two other characterizations of these groups:

- Affine Coxeter groups, whose form B is positive but not definite.
- Groups generated by orthogonal affine reflection and have a certain local finiteness property.

Classification



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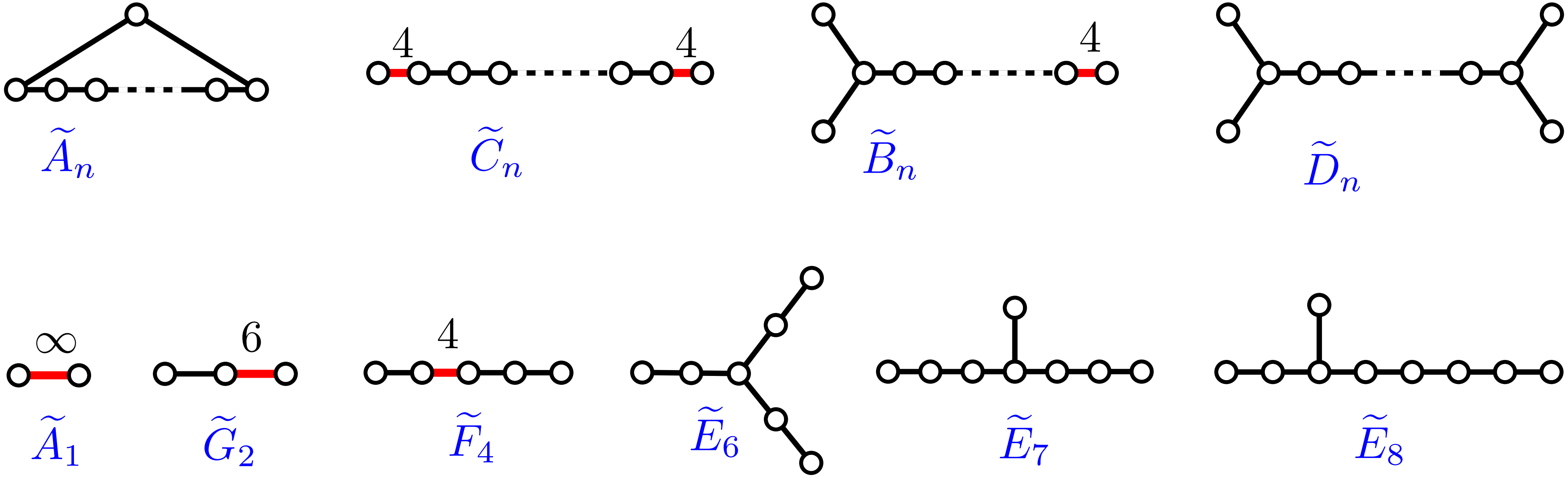

Complements

- The length generating function satisfies

$$W_{aff}(q) = W(q) \prod_{i=1}^{n} \frac{1}{1-q^{e_i}}$$

- By taking any chamber in the complements of reflecting hyperplanes, one obtains a fundamental domain for the action of W_{aff}.

Generalized Noncrossing Partitions

The absolute length

Let W be a finite Coxeter group, and T its set of reflections, i.e. $T = \cup_w wSw^{-1}$.The idea is to take all $t \in T$ as generating set:

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In general there is a nice geometric interpretation of the length: $\ell_T(w)$ is the dimension of the image of $w - 1$ in the geometric representation.

This is based on the nontrivial fact that an expression $t_{\alpha_1} \cdots t_{\alpha_k}$ is reduced if and only if $\{\alpha_1, \ldots, \alpha_k\}$ is linearly independent (Carter's lemma)

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Definition Given $a, b \in W$, define the map
$K_a^b : W \to W, w \mapsto aw^{-1}b$

Proposition K_a^b is an anti-automorphism of the interval $[a, b]$ in the absolute order.

Furthermore $\ell_T(K_a^b(w)) = \ell_T(a) + \ell_T(b) - \ell_T(w)$.

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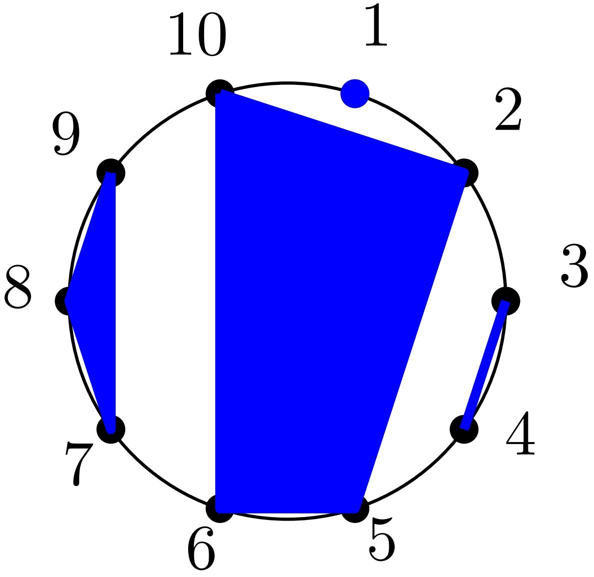

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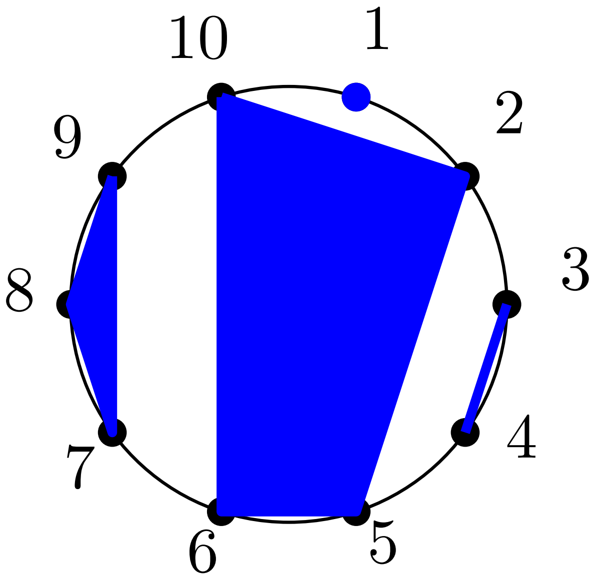

The number of elements of NC_n is the nth Catalan number $\frac{1}{n+1}\binom{2n}{n}$

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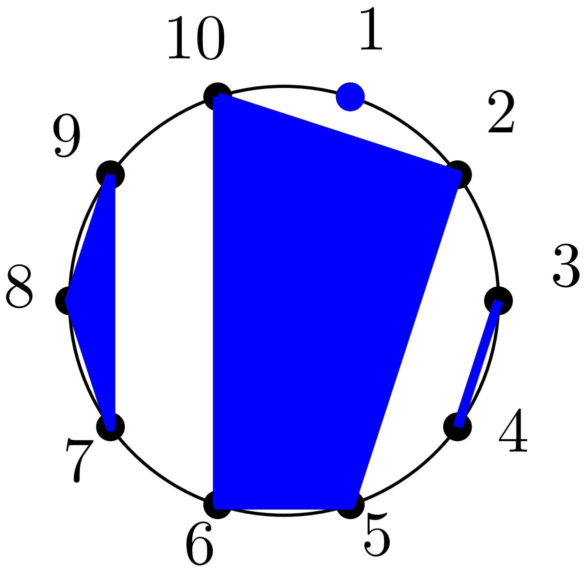

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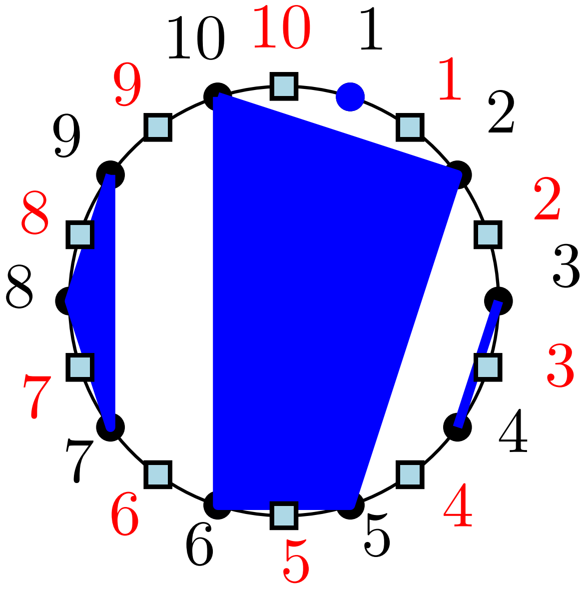

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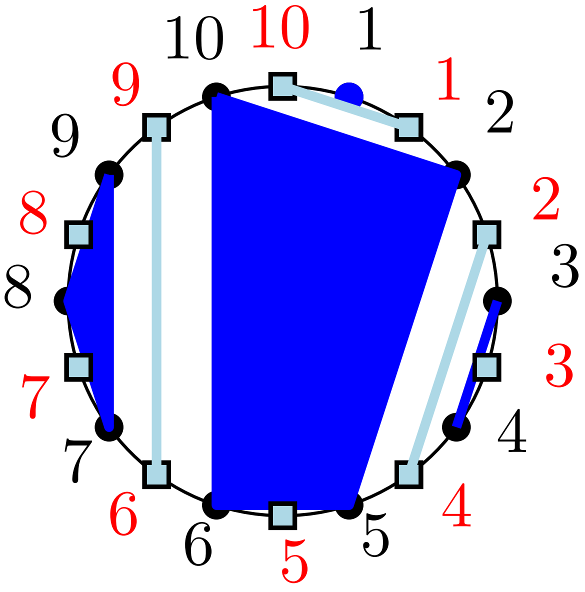

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Theorem For any finite Coxeter group W the cardinality of $NC(W)$ is $\mathrm{Cat(W)}$

Only the case-by-case proof is known.

References

J. Humphreys, *Reflection groups and Coxeter groups*.

A. Björner and F. Brenti, *Combinatorics of Coxeter groups*.

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